Guaranteed Control of Switched Control Systems Using Model Order Reduction and Bisection

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1. CMLA Centre de Mathématiques et de Leurs Applications
2. LMT-Cachan Laboratoire de Mécanique et Technologie
3. LSV Laboratoire de Spécification et Vérification

- Goal : control the evolution of an operating system with the help of actuators
- Framework of the switched control systems : one selects the working modes of the system over time, every mode is described by differential equations (ODEs or PDEs)
- Application to medium/high dimensional systems :
  - Model Order Reduction
  - Error bounding
  - State space bisection

- Described by the differential equation :

$$\begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{cases}$$

  - $x \in \mathbb{R}^n$: state variable
  - $y \in \mathbb{R}^m$: output
  - $u \in \mathbb{R}^p$: control input, takes a finite number of values (modes)
  - A, B, C : matrices of appropriate dimensions
- Idea : impose the right $u(t)$ such that x and y verifies some properties (stability, reachability...)
- Objectives :
  1. *approximate x-stabilization* : make all the state trajectories starting in a compact $R_x \subset \mathbb{R}^n$ return to $R_x + \epsilon_x$;
  2. *guaranteed y-convergence* : send the output of all the trajectories starting in R_x into $R_y \subset \mathbb{R}^m$;

Classical approach of the control theory

Very useful for *on-line* control

- Aim : research of the solution of the dual minimisation problem
- Stability control :
  - Lyapunov method, late 19th
- Optimal control :
  - Hamilton-Jacobi-Bellmann, 1950
  - Lev Pontryagin, 1956
  - Extension to PDEs : J.L.Lions, 1968

- MINIMATOR : code developed at the LSV [L.Fribourg and R.Soulat, 2013]
  - Based on the invariant sets theory
  - Permits the synthesis of *state-dependant* controllers (*correct-by-design*)
  - Operational for ODEs
  - Based on a technique of decomposition of the state-space into local regions where the control is uniform (for a given mode)

Outline

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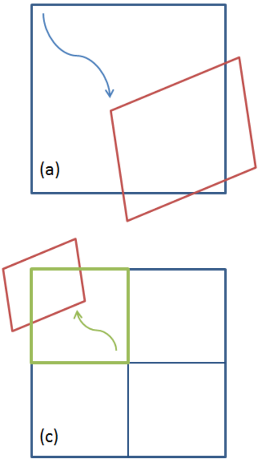
MINIMATOR

- Principle (for the stability) :
  - Let R_x be a region of the state space one wants to control
  - One looks for a pattern (a sequence of control modes) which sends R_x in itself and the output in R_y
  - If no pattern is found, R_x is divided into smaller regions and one looks for patterns which send these sub-regions in R_x and their output in R_y
- Underlying ideas :
  - Temporal discretization
  - Measures carried out at the end of every pattern
  - Linearity of the equations which allows the use of zonotopes (matrices) to represent the regions of the state-space

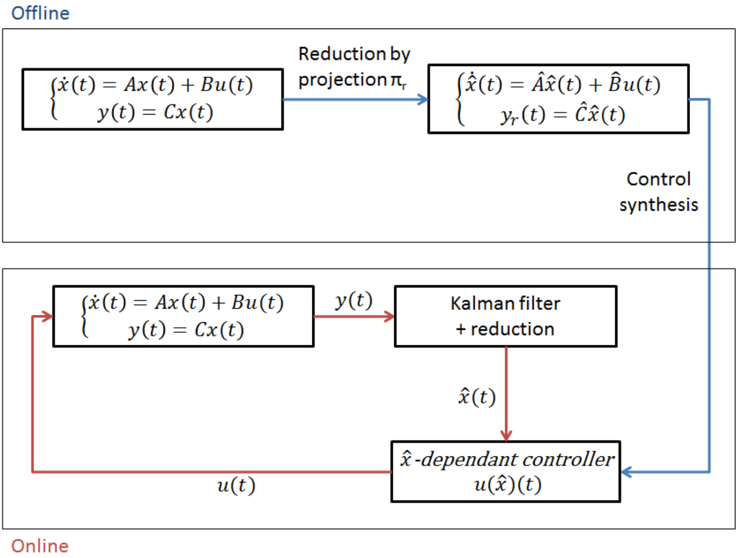
MINIMATOR

- Example : Schematic representation of the box R_x, the sub-boxes and their images



- Properties [L.Fribourg and R.Soulat, 2013] :
  - The convergence properties are proved under limited hypotheses (contracting modes, verified in practice)
  - Computational cost at most in $O(2^{nd}N^k)$
    n : dimension of the state-space
    d : maximal length of decomposition
    N : number of modes
    k : maximal length of the patterns
- Consequence :
  - Dimension of the state-space very limited (< 10 in practice)
  - Necessity of a Model Order Reduction for medium/large scale systems

Procedure of control synthesis



Outline

1. The tool MINIMATOR
2. Model Order Reduction and guaranteed control
3. Numerical results

Gramian based model reduction

The gramians are solutions of the Lyapunov equations :

$$A\mathcal{P} + \mathcal{P}A^* + BB^* = 0_n \tag{1}$$

$$A^*\mathcal{Q} + \mathcal{Q}A + C^*C = 0_n \tag{2}$$

Energetic interpretation :

- The inner product based on $\mathcal{P}^{-1}$ characterizes the minimal energy required to steer the state from the state 0 to x as $t \to \infty$

$$\|x\|^2_{\mathcal{P}^{-1}} = x^\top \mathcal{P}^{-1} x$$

- The inner product based on $\mathcal{Q}$ indicates the maximal energy produced by observing the output of the system corresponding to an initial state x_0 when no input is applied

$$\|x_0\|^2_{\mathcal{Q}} = x_0^\top \mathcal{Q} x_0$$

The balanced truncation

Principle : place ourselves in a basis where both concepts of observability and controllability are equivalent (i.e. such that the system is *balanced*, $\mathcal{P}$ and $\mathcal{Q}$ are equal and diagonal).

Balanced truncation

Computation of a *balancing transformation* T_{bal} and T_{bal}^{-1} :

1. Cholesky : $\mathcal{P} = UU^*$
2. Eigenvalue decomposition : $U^* \mathcal{Q} U = K\Sigma^2 K^*$
3. Computation : $T_{bal} = \Sigma^{1/2} K^* U^{-1}$ and $T_{bal}^{-1} = UK\Sigma^{-1/2}$

The system becomes : $\tilde{A} = T_{bal} A T_{bal}^{-1}, \quad \tilde{B} = T_{bal} B, \quad \tilde{C} = C T_{bal}^{-1}$.
The projection is then : $\pi_r = T_{bal}(:, 1 : n_r)$

Construction of the reduced order system

- Reduction by projection : $\hat{x} = \pi_r x$
- Construction of the reduced system of order n_r :

$$\begin{cases} \dot{\hat{x}} & = \hat{A}\hat{x} + \hat{B}u \\ y_r & = \hat{C}\hat{x} \end{cases} \tag{3}$$

- Associated error :

$$\epsilon_y(x_0, u, t) = \|y(x_0, u, t) - y_r(\pi_r x_0, u, t)\|$$

- Bounded by [4] : $\epsilon_y(t) \le \epsilon^{x_0=0}(t) + \epsilon^{u=0}(t)$ with

$$\epsilon_y^{x_0=0}(t) \le \|u(\cdot)\|_\infty^{[0,t]} \int_0^t \| \begin{bmatrix} C & -\hat{C} \end{bmatrix} \begin{bmatrix} e^{tA} & \\ & e^{t\hat{A}} \end{bmatrix} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \| dt$$

$$\epsilon_y^{u=0}(t) \le \sup_{x_0 \in X_0} \| \begin{bmatrix} C & -\hat{C} \end{bmatrix} \begin{bmatrix} e^{tA} & \\ & e^{t\hat{A}} \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_r x_0 \end{bmatrix} \|$$

- Computation of the bound :

$$\epsilon_y = \sup_{j \in [\![1,\ldots,+\infty[\![} \left\{ \|u(\cdot)\|_{\infty}^{[0,j\tau]} \int_0^{j\tau} \left\| \begin{bmatrix} C & -\hat{C} \end{bmatrix} \begin{bmatrix} e^{tA} & \\ & e^{t\hat{A}} \end{bmatrix} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \right\| dt \quad + \right.$$

$$\left. \sup_{x_0 \in X_0} \left\| \begin{bmatrix} C & -\hat{C} \end{bmatrix} \begin{bmatrix} e^{j\tau A} & \\ & e^{j\tau \hat{A}} \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_R x_0 \end{bmatrix} \right\| \right\}$$

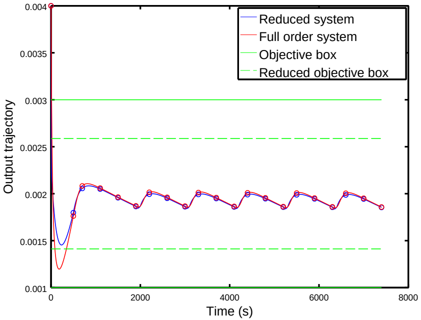
- Objectives for the full order system : R_x, R_y.
- Control synthesis on the reduced system with the objectives : $\hat{R}_x = \pi_r R_x$ and R_y^{ϵ} defined as :

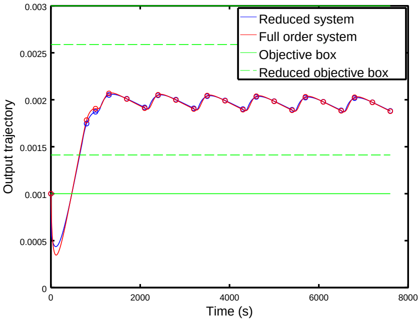
$$R_y^{\epsilon} := \text{sub-box of} \quad R_y \quad \text{such that :} \quad \forall y \in \partial R_y, \forall y' \in R_y^{\epsilon}, \|y - y'\| \geq \epsilon_y^{\infty}$$

Outline

Offline control

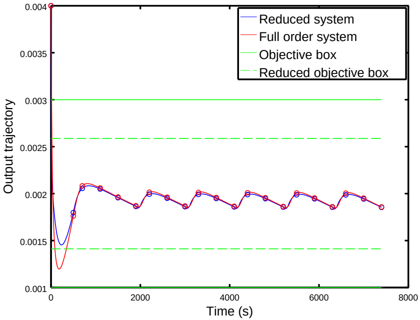
- Computation of a pattern sequence with the reduced system, direct application on the full order system
- Test on a linearized model of a distillation column [5] : $n = 11$ and $n_r = 2$

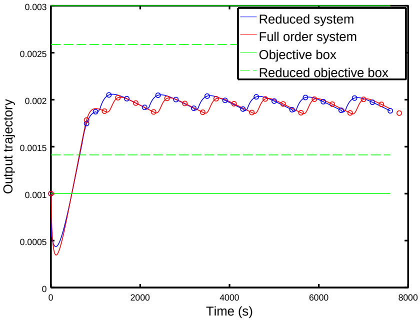




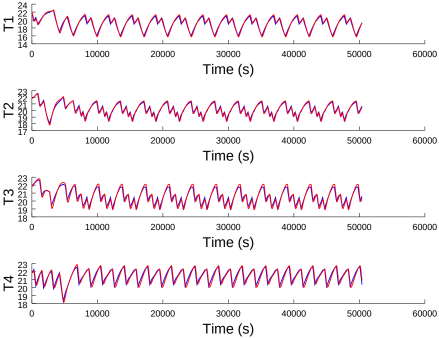
Online control

- Having synthesized (offline) the controller :
  1. Unknown initial state : x_0
  2. From the output y_0, reconstruction of the reduced state $\hat{x}_0$
  3. Application of the pattern $u(\hat{x}_0)$, the state is sent to x_1
  4. Unknown initial state x_1...
- Current work : exact reconstruction





- Control of the temperature of a 4 room appartment : offline and online first tests



Open questions and future work

- Remaining issue :
  - guaranteeing that the projection of the full order system state $\pi_r x$ stays in $\hat{R}_x$
    $\Rightarrow$ guaranteed x-stabilization
    $\Rightarrow$ projection on the border of $\hat{R}_x$
- Future work :
  - Online reconstruction of the reduced state
    $\Rightarrow$ reduced Kalman filter
    $\Rightarrow$ reconstruction error estimation
  - Application to large scale systems

Some references

[1] Peter Benner and André Schneider. Balanced truncation model order reduction for lti systems with many inputs or outputs. In *Proceedings of the 19th international symposium on mathematical theory of networks and systems–MTNS*, volume 5, 2010.

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[5] D. Tong, W. Zhou, A. Dai, H. Wang, X. Mou, and Y. Xu. $\mathcal{H}_\infty$ model reduction for the distillation column linear system. *Circuits Syst Signal Process*, 2014.